Lecture 2
Vector analysis and calculus

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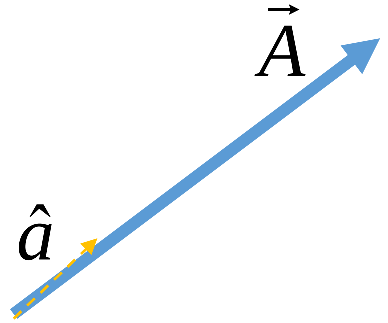
Vectors and Scalars

Vector : has both magnitude and direction in space as velocity, force, electric field

$$\vec{A} = \left|\vec{A}\right| \hat{a} = A\,\hat{a} = A_x(x,y,z)\,\hat{a}_x + A_y(x,y,z)\,\hat{a}_y + A_z(x,y,z)\,\hat{a}_z$$

$$\left|\vec{A}\right| = A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad \hat{a} = \frac{\vec{A}}{\left|\vec{A}\right|}, \quad \left|\hat{a}\right| = 1$$



Scalar quantity : has magnitude only as temperature, mass, volume,….etc.

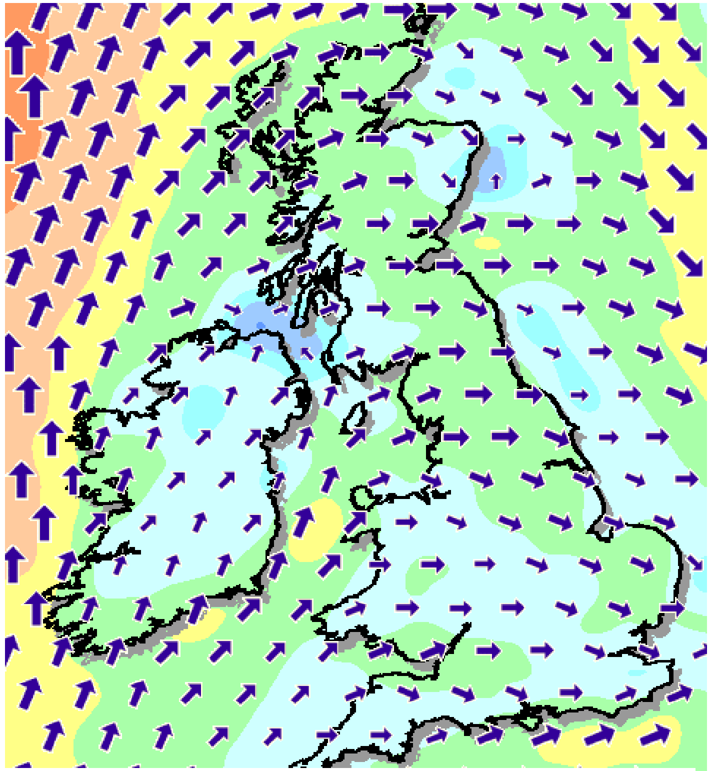
$$\vec{E} = E\hat{a}_x$$

Vector scalar unit vector

$\overline{E}, \underline{E}$

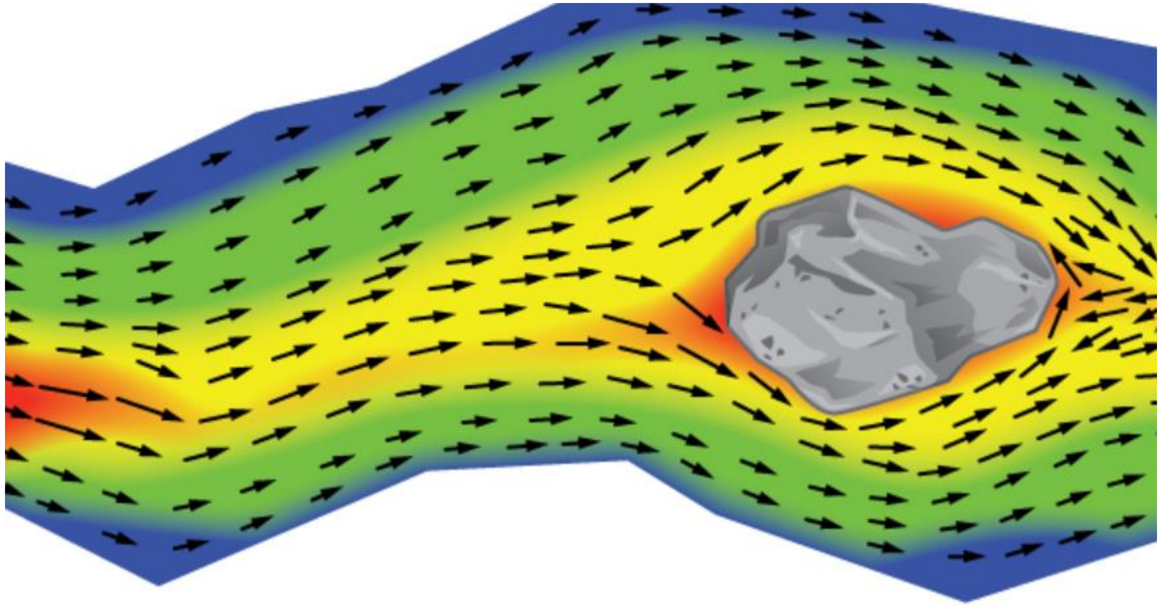
Other notation

vector field examples



wind map

every point has a vector has magnitude (velocity of wind) and direction (of velocity).



water flow

The vector velocity field of water on the surface of a river shows the varied speeds of water.

The Cartesian Coordinate System

$$source\ \vec{A} = (x_1, y_1, z_1),$$

$$observation\ point\ \vec{B} = (x_2, y_2, z_2)$$

$$\vec{A} + \vec{R} = \vec{B} \rightarrow \vec{R} = \vec{B} - \vec{A}$$

$$\vec{R} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

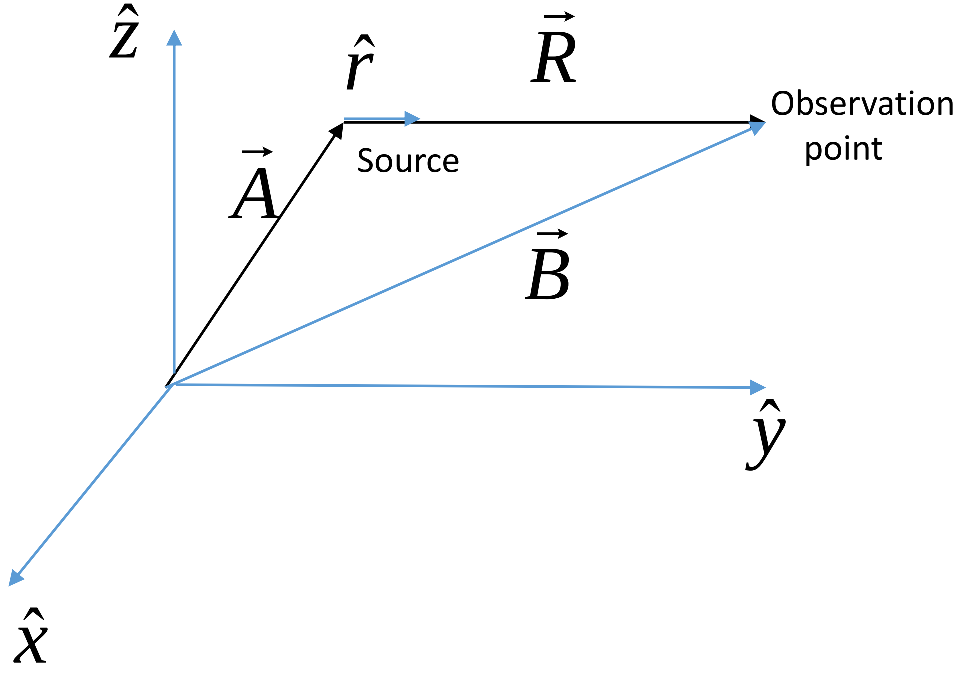
$$= (x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y} + (z_2 - z_1)\hat{z}$$

$$distance\ between\ source\ and\ observation\ point$$

$$\left|\vec{R}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad , \vec{R} = \left|\vec{R}\right|\hat{r}$$

Direction from source
To observation point

Distance between source
and observation point



Example 1.1

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$.

Solution. the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

the magnitude of $\mathbf{G}$,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

the unit vector

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

D1.1. Given points $M(-1, 2, 1)$, $N(3, -3, 0)$, and $P(-2, -3, -4)$, find: (*a*) $\mathbf{R}_{MN}$; (*b*) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (*c*) $|\mathbf{r}_M|$; (*d*) $\mathbf{a}_{MP}$

Ans. $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$; $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$; 2.45; $-0.1400\mathbf{a}_x - 0.700\mathbf{a}_y - 0.700\mathbf{a}_z$

Vector Multiplication:

The dot Product(Scalar Product)

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$ ← scalar

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$ the result is scalar

If: $$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$

$$\vec{B}. = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z \rightarrow \vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Notes:

Vector dotted with itself is magnitude square

$$\vec{A}.\vec{A} = |\vec{A}|^2 = A^2$$

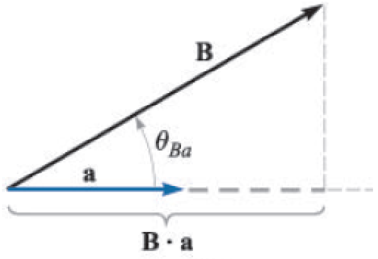
any unit vector dotted with itself is unity

$$\hat{a}_x.\hat{a}_x = 1$$

dot product find the component of vector in a givin direction

$$\vec{B}.\hat{a} = |\vec{B}|\cos\theta_{Ba}$$

Unit vector



The Cross Product(Vector Product)

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta \quad \hat{n}$$

Result is vector perpendicular to A&B

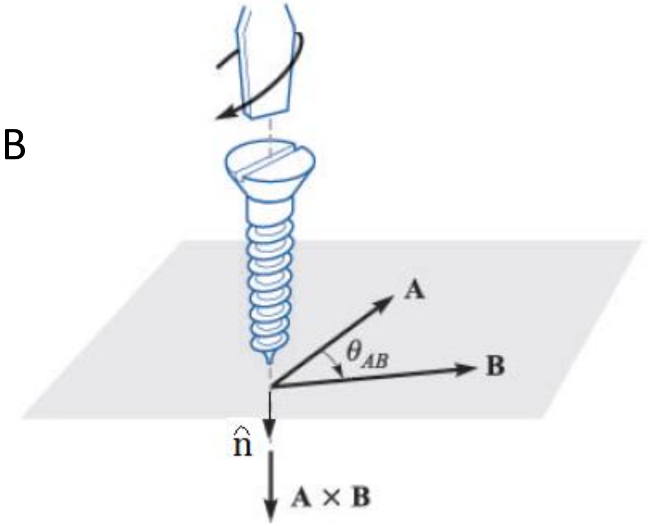


FIGURE 1.5
The direction of **A** × **B** is in the direction of advance of a right-handed screw as **A** is turned into **B**.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Notes:

$$\hat{a}_x\times\hat{a}_y = \hat{a}_z,\ \hat{a}_y\times\hat{a}_z = \hat{a}_x,\ \hat{a}_z\times\hat{a}_x = \hat{a}_y$$

$$\hat{a}_x\times\hat{a}_x = \hat{a}_y\times\hat{a}_y = \hat{a}_z\times\hat{a}_z = 0$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}, \quad \vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

Other Coordinate Systems: Circular and cylindrical coordinates

Cylindrical coordinates (ρ,ϕ,z)

$$\overline{P} = \rho_1 \hat{a}_\rho + \phi_1 \hat{a}_\phi + z_1 \hat{a}_z$$

variation:

$0 \le \rho < \infty,\ 0 \le \phi \le 2\pi,\quad -\infty \le Z \le \infty$

Differential length:

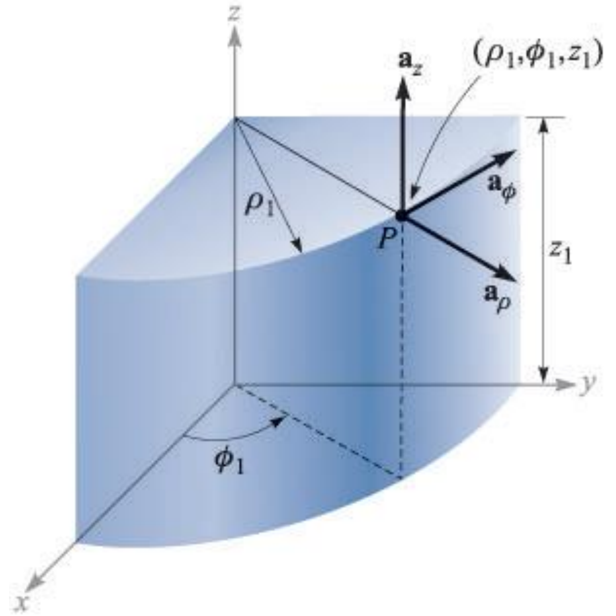
$$d\vec{l} = d\rho\,\hat{a}_\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$

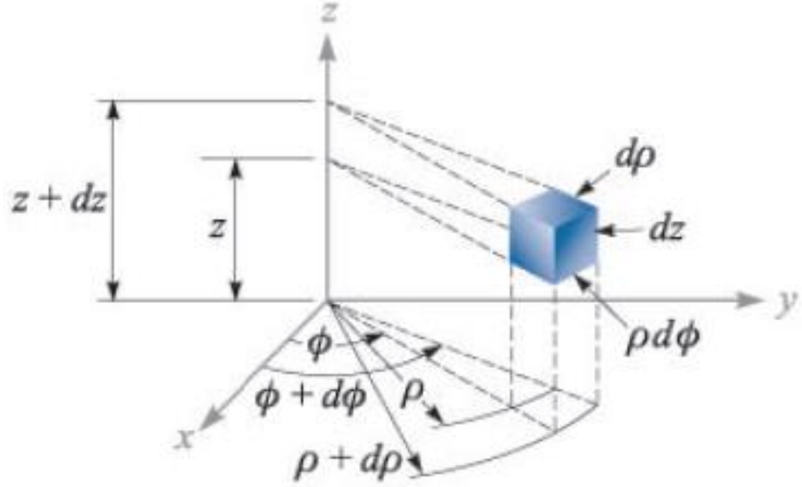
Differential area:

$$d\overline{S} = \rho d\phi dz\,\hat{a}_\rho + d\rho dz\,\hat{a}_\phi + \rho d\rho d\phi\,\hat{a}_z$$

Differential Volume:

$$dv = \rho d\rho d\phi dz$$





Relation between Cylindrical and cartesian coordinates

If I have ρ, φ and z

X= $\rho\cos\varphi$

y= $\rho\sin\varphi$

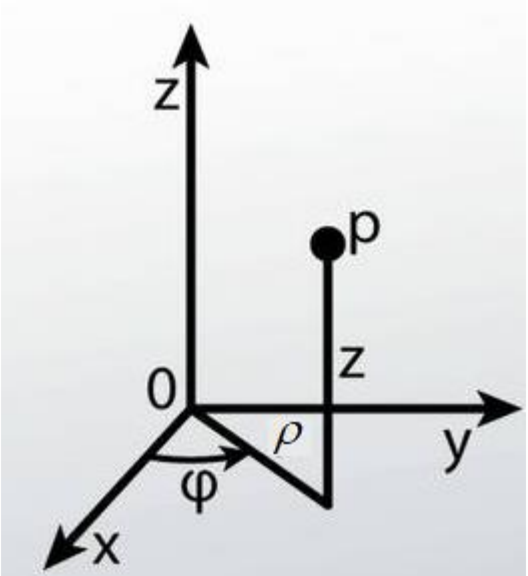
Z=Z

If I have x, y and z

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$Z = Z$$



Example

Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

Solution. The new components are

$$B_\rho = \mathbf{B}\cdot\mathbf{a}_\rho = y(\mathbf{a}_x\cdot\mathbf{a}_\rho) - x(\mathbf{a}_y\cdot\mathbf{a}_\rho)$$
$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$
$$B_\phi = \mathbf{B}\cdot\mathbf{a}_\phi = y(\mathbf{a}_x\cdot\mathbf{a}_\phi) - x(\mathbf{a}_y\cdot\mathbf{a}_\phi)$$
$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

Thus,

$$\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$$

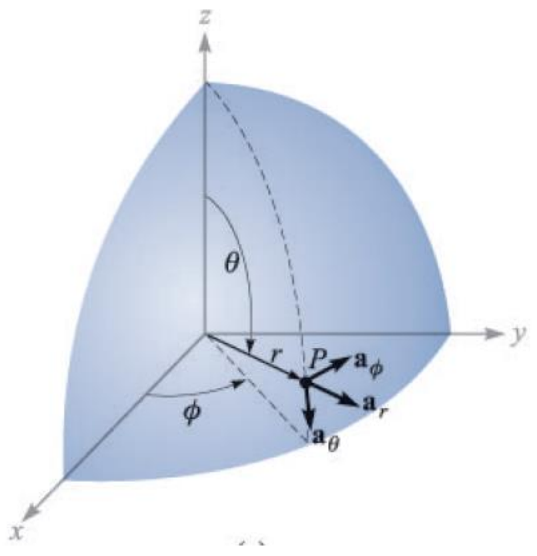
	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{a}_\rho$	cos φ	sin φ	0
$\hat{a}_\phi$	-sin φ	cos φ	0
$\hat{a}_z$	0	0	1

Spherical coordinates (r, θ ,ϕ)

$$\overline{P} = P_r\hat{a}_r + P_\theta\hat{a}_\theta + P_\phi\hat{a}_\phi$$

variation:

$0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$



Differential length:

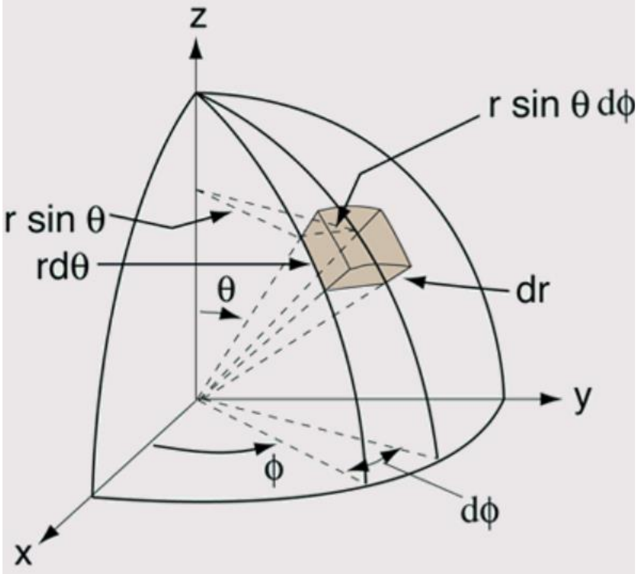
$$d\vec{l} = dr\ \hat{a}_r + rd\theta\ \hat{a}_\theta + r\sin\theta d\phi\ \hat{a}_\phi$$

Differential area:

$$d\overline{S} = r^2\sin\theta d\theta d\phi\ \hat{a}_r + r\sin\theta drd\phi\ \hat{a}_\theta + rd\theta dr\ \hat{a}_\phi$$

Differential Volume:

$$dv = r^2\sin\theta drd\theta d\phi$$



Relation between spherical and cartesian coordinates

If I have r, θ and φ

X=rsinθcosφ

Y=rsinθsinφ

Z=rcos θ

x,y,z

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

Example

We illustrate this transformation procedure by transforming the vector field $\mathbf{G} = (xz/y)\mathbf{a}_x$ into spherical components and variables.

Solution.

$$G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi$$

$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi$$

$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi)$$

$$= -r\cos\theta\cos\phi$$

Collecting these results, we have

$$\mathbf{G} = r\cos\theta\cos\phi(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$

	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$
$\mathbf{a}_r$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
$\mathbf{a}_\phi$	$-\sin\phi$	$\cos\phi$	0